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SOME SOLUTIONS OF THE FINITE SPECTRUM PROBLEM AND APPLICATIONS

SOFIANE MESSIRDI[1], MOHAND OULD ALI[2] AND BEKKAI MESSIRDI[3*]

Abstract. In this paper, we characterize linear operators with finite spectrum in the Banach spaces setting via quasi-Fredholm and pseudo-Fredholm spectrums. As applications, we find sufficient conditions under which the spectrum of operator matrices and that of totally paranormal operators is finite.

1. Introduction and Preliminaries

When working in PDE then, one is often times forced to consider unbounded operators between Banach spaces, while such operators are not continuous. Almost every differential operators we encounter in practice are closed and densely defined (with dense domain). Let X be a Banach space, we denote by $\mathcal{B}(X)$ (resp. $\mathcal{C}(X)$) the set of all bounded (resp. closed, densely defined) linear operators from X into X. For $T \in \mathcal{C}(X)$, we write $D(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset X$ for the range of T. Let $\sigma(T)$ denote the spectrum of T and I be the identity operator in X. The restriction of T to a subspace M of X is denoted by $T_{|M}$. $\overline{U}$ denotes the closure of a subset U of X.

Let T be a linear operator in X, which is not necessarily everywhere defined. It is well known that $N(T^n) \subset N(T^{n+1})$ and $R(T^{n+1}) \subset R(T^n)$ for all $n \in \mathbb{N}$. The smallest nonnegative integer for which there is equality is called the ascent of T and the descent of T, denoted by $a(T)$ and $d(T)$ respectively. In case no such number exists the ascent or descent of T is said to be infinite. The spectrum of a linear operator on a finite-dimensional space over an arbitrary field is the finite set of its eigenvalues. The problem of classifying linear operators with finite spectrum on infinite-dimensional spaces is largely open, very little works are known about this question. Aupetit proved in [4] that $\frac{A}{Rad(A)}$ is finite dimensional, for a complex Banach algebra A and where $Rad(A)$ is the radical of A, if the spectrum of every element of this algebra is finite. We know, on the other hand, that a normal operator in the Calkin algebra is the norm limit of a sequence of normal operators with finite spectra if and only if its index function is zero everywhere on its domain (see Lin, [Theorem 4.4, [15]]). On the other hand, an operator is called spectral if it has a resolution of the identity. Quasinilpotent operators are spectral, more generally every operator with a finite spectrum is spectral. Recall that a bounded operator $T \in \mathcal{B}(X)$ is said to be algebraic if there exists a non-constant polynomial P such that $P(T) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that every finite-dimensional operator is algebraic. It is also clear, by virtue of

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Corresponding author.

spectral mapping theorem, that every algebraic operator has a finite spectrum. In fact, it is a non-exhaustive list of operators with finite spectrum.

A densely defined closed linear operator T on X is said to be a Fredholm operator if $\dim N(T)$ and $\dim \frac{X}{R(T)}$ are finite and $R(T)$ is closed in X. Operators on X such that $\dim N(T)$ or $\dim \frac{X}{R(T)}$ is finite and $R(T)$ is closed in X are semi-Fredholm operators. Let $\Phi(X)$ denote the set of Fredholm operators. $T \in \mathcal{B}(X)$ is called a Riesz operator if $(\lambda I - T) \in \Phi(X)$ for all scalars $\lambda \neq 0$. The Riesz operators on a Banach space have a spectral theory like that of compact operators, indeed the spectrum of a Riesz operator is finite or a sequence which clusters at 0 and that every Riesz operator on a Hilbert space can be decomposed into the sum of compact and quasinilpotent parts. One of the most important classes is that of semi-regular operators, $T \in \mathcal{C}(X)$ is said to be semi-regular if $R(T)$ is closed and $N(T^n) \subseteq R(T)$, for all integer $n \geq 0$. In particular, if T is semi-regular then T^n is semi-regular for every $n \in \mathbb{N}$ and if $R^\infty(T) = \bigcap_{n=0}^{\infty} R(T^n)$, $T(R^\infty(T) \cap D(T)) = R^\infty(T)$ and $R^\infty(T)$ is closed in X.

Let (M, N) be a pair of closed subspaces of X. A linear operator T on X is said to be completely reduced by the pair (M, N) if $X = M \oplus N$ and

$$PD(A) \subset D(A), \ TM \subset M \text{ and } TN \subset N,$$

where P is the projection on M along N. When T is completely reduced by (M, N) as above, the pairs $T_{|M}$, $T_{|N}$ of T in M and N respectively can be defined, $T(M \cap D(T)) \subseteq M$, $N \subseteq D(T)$, $T_{|M}$ is the linear operator defined on M with $D(T_{|M}) = D(T) \cap M$ such that $T_{|M}x = Tx \in M$, $T_{|N}$ is defined similarly. In this case we write $T = T_{|M} \oplus T_{|N}$. By using Lemma 8.1, Theorems 8.2 and 8.3 of Sandovici et al. [18], it is simple to check the following properties:

Lemma 1.1. *Let $T \in \mathcal{C}(X)$ be completely reduced by the pair (M, N). Then*

1) $N(T) = N\left(T_{|M}\right) \oplus N\left(T_{|N}\right)$ and $R(T) = R\left(T_{|M}\right) \oplus R\left(T_{|N}\right)$.

2) For all $p \in \mathbb{N}$, T^p is completely reduced by the pair (M, N) and $T^p = T^p_{|M} \oplus T^p_{|N}$.

3) $a(T) < \infty$ if and only if $a(T_{|M})$ and $a(T_{|N})$ are both finite. Similar property for $d(T)$.

4) If $X = R(T^p) \oplus N(T^p)$ for some nonnegative integer p, then $a(T) = d(T) \leq p$, and T is completely reduced by the pair $(R(T^p), N(T^p))$.

$T \in \mathcal{C}(X)$ is called of Kato (respectively, Generalized Kato) type, if there exists a pair of T-invariant closed subspaces (M, N) of X such that $X = M \oplus N$, $T = T_{|M} \oplus T_{|N}$ where $T_{|M}$ is semi-regular and $T_{|N}$ is nilpotent (respectively, quasinilpotent). Generalized Kato type operators are also called pseudo-Fredholm operators and the pair (M, N) is called the generalized Kato decomposition of the linear operator T, GKD for short. Clearly, a closed semi-Fredholm operator is of Kato type, furthermore every semi-regular operator is of Kato type with $M = X$ and $N = \{0\}$ and a quasinilpotent operator has a GKD with $M = \{0\}$ and $N = X$. Examples of pseudo-Fredholm operators are Kato type operators, semi-regular operators, semi-Fredholm operators and quasi-Fredholm operators. If 0 is an isolated point in the spectrum $\sigma(T)$ of a bounded operator T, then T is pseudo-Fredholm. If (M, N) is a GKD of T then $(N^\perp, M^\perp)$ is a GKD of T^* adjoint operator of T. Recall that $T \in \mathcal{C}(X)$ is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that:

1) $R(T^n) \cap N(T) = R(T^d) \cap N(T)$ for all $n \geq d$;

2) $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are closed in X.

An operator is quasi-Fredholm if it is quasi-Fredholm of some degree d. Semi-regular operators (quasi-Fredholm of degree $d = 0$), surjective operators as well as injective operators with closed range, Fredholm operators, semi-Fredholm operators and B-Fredholm operators are quasi-Fredholm operators. Labrousse [14] studied and characterized the class of quasi-Fredholm operators, in the case of Hilbert spaces, and he prove that this class coincides with the set of Kato type operators. But in the case of Banach spaces the Kato type operator is also quasi-Fredholm, the converse is not true. The following example shows that the class of quasi-Fredholm operators is a proper subclass of pseudo-Fredholm operators.

Example 1.2. ([17]) Let H be a separable Hilbert space with an orthonormal basis $(e_{i,j})$, where i and j are integers such that $ij \leq 0$. Define the operator $T \in \mathcal{B}(H)$ by:

$$Te_{i,j} = \begin{cases} 0 \text{ if } i = 0,\ j > 0 \\ e_{i+1,j} \text{ otherwise} \end{cases}$$

$N(T) = \bigcup\limits_{j \geq 1} \{e_{0,j}\} \subset R^\infty(T)$ and $R(T)$ is closed. Then T is semi-regular and thus a pseudo-Fredholm operator. Let Q a quasinilpotent operator in H which is not nilpotent and no commute with T, then $S = T \oplus Q$ is a pseudo-Fredholm operator but is not Kato type.

Müller has shown in his paper [16] that a bounded quasi-Fredholm operator on a Banach space remains of Kato type provided to add an hypothesis of complementation. Let us also point out that this was mentioned earlier by Labrousse in [14]. We present this result in the context of closed densely defined operators where the proofs can be used practically without modification with a particular attention to be paid to the domain of the operator.

Theorem 1.3. *Let $T \in \mathcal{C}(X)$ be a quasi-Fredholm of degree $d \in \mathbb{N}$ such that such the subspaces $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are complemented in X. Then T is of Kato type, more precisely there are closed subspaces M and N such that $X = M \oplus N$, $T(M \cap D(T)) \subseteq M$, $N \subseteq D(T)$, $TN \subseteq N$, where $T_{|M}$ is semi-regular and $T_{|N}$ is nilpotent with degree d.*

For $T \in \mathcal{C}(X)$, let us define the essential quasi-Fredholm and pseudo-Fredholm spectrum as follows respectively:

$$\sigma_{qF}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not quasi-Fredholm}\},$$

$$\sigma_{pF}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not pseudo-Fredholm}\}.$$

It is well known that $\sigma_{qF}(T)$ and $\sigma_{pF}(T)$ are closed subsets of the spectra $\sigma(T)$ of $T \in \mathcal{B}(X)$, and they may be empty. For example, all quasinilpotent operator has an empty pseudo-Fredholm spectrum.

The aim of this paper is the characterization of linear operators with finite spectrum on a Banach space specially via the pseudo-Fredholm spectrum. We prove that if the pseudo-Fredholm essential spectrum is empty, then the spectrum of the operator is finite. This result constitutes a generalization of Propositions 5.1.1 of [14]. Finally, we prove as application that if $\sigma_{pF}(A)$ and $\sigma_{pF}(B)$ are empty, then $\sigma(M_C)$ is finite, for every $C \in \mathcal{B}(X, Y)$ where $M_C = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, $A \in \mathcal{B}(X)$ and

$B \in \mathcal{B}(Y)$, X and Y are Banach spaces. We also establish a new result on the finite spectrum of a totally paranormal operator on a Hilbert space.

Our paper is organized as follows: In sections 1 and 2, we give some definitions and preliminary results in which our investigation will be done. We present also some interesting results on the finite spectrum problem in a more general context that of closed operators via quasi-Fredholm and pseudo-Fredholm spectrums. Finally, in section 3, we apply the obtained results to operator matrices and totally paranormal operators.

2. Some preliminary and main results

In this section, we collect some technical results which we will need use in the sequel and that provide some answers to the problem of finite spectrum. If $\lambda_0 \in \sigma(T)$, $T \in \mathcal{C}(X)$, then $0 < a(\lambda_0 I - T) = d(\lambda_0 I - T) < \infty$ if and only if λ_0 is a pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$, in this case λ_0 is an isolated point of $\sigma(T)$ (see [9]). In the following, we show that an operator with finite descent (respectively, with dense range) is pseudo-Fredholm if and only if it is of Kato type (respectively, semi-regular).

Theorem 2.1. *Let $T \in \mathcal{C}(X)$.*

1) If $d(T) < \infty$, then T is pseudo-Fredholm if and only if T is of Kato type.

2) If $\overline{R(T)} = X$, then T is pseudo-Fredholm if and only if T is semi-regular.

Proof. 1) If T is of Kato type then T is pseudo-Fredholm. Conversely, if T is a pseudo-Fredholm then there exists a pair of T-invariant closed subspaces (M, N) of X such that $X = M \oplus N$, $T = T_{|M} \oplus T_{|N}$ where $T_{|M}$ is semi-regular and $T_{|N}$ is quasinilpotent. $d(T) < \infty$, implies that $d(T_{|M}) < \infty$ and $d(T_{|N}) < \infty$, which shows in particular that $T_{|N}$ is nilpotent. Thus, T is a Kato type operator.

2) If T is semi-regular then T is necessarily pseudo-Fredholm. Conversely, if T admits a GKD (M, N) such that $T = T_{|M} \oplus T_{|N}$, $T_{|M}$ is semi-regular and $T_{|N}$ is quasinilpotent. As $N(T^*) = \{0\}$, we deduce that $K(T) = M$. Since M is closed and $T_{|M}$ is semi-regular, then $M = X$ therefore T is semi-regular on X. □

Theorem 2.2. *Let $T \in \mathcal{B}(X)$. Then $\sigma(T)$ is a finite set of poles of the resolvent $R(\lambda, T)$ if and only if $d(\lambda_0 I - T) < \infty$, for every $\lambda_0 \in \mathbb{C}$.*

Proof. If $\sigma(T)$ is a finite set of poles of the resolvent $R(\lambda, T)$, then $d(\lambda_0 I - T) < \infty$ for every $\lambda_0 \in \mathbb{C}$. Conversely, if $d(\lambda_0 I - T) < \infty$ for every $\lambda_0 \in \mathbb{C}$, then $d(\lambda_0 I - T) < \infty$ for all $\lambda_0 \in \partial\sigma(T)$, $\partial\sigma(T)$ the boundary of $\sigma(T)$, which implies that every $\lambda_0 \in \partial\sigma(T)$ is a pole of the resolvent $R(\lambda, T)$ and $\partial\sigma(T) = \sigma(T)$ ([19]). So $\sigma(T)$ is a finite set of poles of the resolvent $R(\lambda, T)$. □

The notion of pseudo-Fredholm operators is strongly attached to the analytical core and the quasinilpotent part of a linear operator. The analytical core $K(T)$ of $T \in \mathcal{C}(X)$ is defined by ([21]):

$$\begin{aligned} K(T) &= \{x \in X : \text{ there exist a sequence } (x_n) \text{ in } X \text{ and a constant } \delta > 0 \text{ such that} \\ x_0 &= x, Tx_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n \in \mathbb{N}\} \end{aligned}$$

and the quasinilpotent part $H_0(T)$ of T is defined by:

$$H_0(T) = \left\{ x \in X : \lim_{n \to \infty} \|T^n x\|^{1/n} = 0 \right\}.$$

We note that $H_0(T)$ and $K(T)$ are generally non closed subspaces of X and $T(K(T)\cap D(T)) = K(T)$, $H_0(T) \subseteq D(T)$ and $T(H_0(T)) \subseteq H_0(T)$. If X' is a closed subspace of X such that $T(X') = X'$, then $X' \subset K(T)$. Furthermore, if $T \in \mathcal{B}(X)$ is quasinilpotent then $K(T) = K(T^*) = \{0\}$. If $T \in \mathcal{C}(X)$, then $H_0(T) = X$ if and only if T is quasinilpotent.

If T is semi-regular then $K(T) = R^\infty(T)$, $\overline{H_0(T)} = \overline{\bigcup_{k\geq 0} N(T^k)}$ and $T(\overline{H_0(T)}) = \overline{H_0(T)}$. On the other hand, it is well known that if $T \in \mathcal{C}(X)$ then the following conditions are equivalent:

(1) λ is an isolated point in the spectrum $\sigma(T)$ of T;

(2) $K(\lambda I - T)$ and $H_0(\lambda I - T)$ are closed, T-invariant subspaces of X and $X = K(\lambda I - T) \oplus H_0(\lambda I - T)$.

See [1] for more informations about $K(T)$ and $H_0(T)$. In the paper [8], Gong and Wang show for a compact operator T on X the following equivalences:

$$\begin{aligned} K(T) \text{ is closed in } X &\iff 0 \text{ is an isolated point of } \sigma(T).\\ &\iff \dim K(T) < \infty.\\ &\iff \sigma(T) \text{ is a finite set.}\end{aligned}$$

Using similar demonstrations, Bouamama has verified in [6] that these equivalences are also true for Riesz operators and more generally for meromorphic operators. $T \in \mathcal{B}(X)$ is called meromorphic if the spectrum $\sigma(T)$ of T is a countable set, with 0 the only possible accumulation point, such that all the nonzero points of $\sigma(T)$ are poles of the resolvent $R(\lambda, T)$ which is equivalent to $\max(a(\lambda I - T), d(\lambda I - T)) < \infty$ for all $\lambda \in \mathbb{C}\setminus\{0\}$. It is clear that if T is a Riesz operator then T is meromorphic. But, a meromorphic operator is not always Riesz operator, it suffices, for example, to consider a bounded projection P whose kernel is of infinite dimension, so $a(P) = d(P) < \infty$, and P is not a Riesz operator.

Theorem 2.3. *([6]) Let $T \in \mathcal{B}(X)$ be meromorphic.Then the following conditions are equivalent:*

1) $K(T)$ is closed.

2) 0 is an isolated point of $\sigma(T)$.

2) $T = Q + F$, whre $Q, F \in \mathcal{B}(X)$, $QF = FQ = 0$, Q is quasinilpotent and $\sigma(F)$ is a finite set.

3) $\dim \frac{X}{H_0(T)} < \infty$.

4) $\sigma(T)$ is a finite set.

For pseudo-Fredholm operators we have the following properties on $K(T)$ and $H_0(T)$ easily generalizable to closed operators on a Banach space.

Theorem 2.4. *(see [3]),[7]) Let $T \in \mathcal{C}(X)$ be pseudo-Fredholm with a GKD (M, N). Then,*

1) $K(T) = K(T_{|M}) = R^\infty(T_{|M})$ is closed.

2) $K(T) \cap N(T) = N(T_{|M})$.

3) $R(T) + H_0(T) = R(T_{|M}) \oplus N$ is closed.

3) $H_0(T) = H_0(T_{|M}) \oplus H_0(T_{|N}) = H_0(T_{|M}) \oplus N$.

4) T^* is pseudo-Fredholm with a GKD $(N^\perp, M^\perp)$.

On the other hand,

$$\lambda \text{ is an isolated point in } \sigma(T) \iff \lambda \in \partial\sigma(T) \cap \sigma_{pF}(T)$$

and $\sigma_{pF}(T)$ is at most countable if and only if $\sigma(T)$ is at most countable. It is also shown in the paper [11] that the generalized Kato spectrum of a bounded operator is a closed subset of the spectra $\sigma(T)$ of T and that $\sigma_K(T)\diagdown\sigma_{pF}(T)$ consist of at most countably many isolated points, where $\sigma_K(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not Kato type$\}$. See [5] for more informations about $\sigma_{pF}(T)$.

The next theorem shows that the pseudo-Fredholm spectrum of a closed densely defined operator T on X is a closed subset of the spectra $\sigma(T)$ of T and generalizes Theorem 2.2 of [11] to unbounded operators. The proof of this result uses the notion of reduced minimum modulus of a linear operator which we recall here briefly. If $T \in \mathcal{C}(X)$, we denote by $C(T) = D(T) \cap N(T)^{\perp}$ the carrier of T and the operator $T_{|C(T)}$ by T_0. The reduced minimum modulus of T is defined by:

$$\gamma(T) = \inf\{\|Tx\| : x \in C(T),\ \ \|x\| = 1\}.$$

Some known properties of $\gamma(T)$ which are listed in the following proposition are used here.

Proposition 2.5. *(*[13]*) For a densely defined operator $T \in \mathcal{C}(X)$, the following statements are equivalent.*
(1) $R(T)$ is closed.
(2) $R(T^*)$ is closed.
(3) $T_0 = T_{|C(T)}$ has a bounded inverse.
(4) $\gamma(T) = \gamma(T^*) > 0$.

Theorem 2.6. *Suppose that $T \in \mathcal{C}(X)$ is pseudo-Fredholm, then there exists a constant $\epsilon > 0$ such that for all $\lambda \in D(0,\epsilon)\diagdown\{0\}$, $(\lambda I - T)$ is semi-regular, where $D(0,\epsilon)$ is the open disc around zero of radius $\epsilon > 0$.*

Proof. If $T \in \mathcal{C}(X)$ admits a generalized Kato decomposition (M, N) of T-invariant closed subspaces (M, N) of X such that $X = M \oplus N$, $T = T_{|M} \oplus T_{|N}$ where $T_{|M\cap D(T)}$ is semi-regular and $T_{|N}$ is quasinilpotent. If $M = \{0\}$, T is quasinilpotent or $\sigma(T) = \{0\}$, then for all $\lambda \in \mathbb{C}\diagdown\{0\}$, $(\lambda I - T)$ is boundedly invertible and so $\lambda I - T$ is semi-regular. If $M \neq \{0\}$, T admits the block matrix representation:

$$T = \begin{pmatrix} T_{|M\cap D(T)} & 0 \\ 0 & T_{|N} \end{pmatrix}$$

with respect to $X = M \oplus N$. So

$$\lambda I - T = \begin{pmatrix} (\lambda I - T)_{|M\cap D(T)} & 0 \\ 0 & (\lambda I - T)_{|N} \end{pmatrix}.$$

Since $T_{|M\cap D(T)}$ is semi-regular, then $R\left(T_{|M\cap D(T)}\right)$ is closed and we have $\gamma\left(T_{|M\cap D(T)}\right) > 0$. By using Theorem 1.31 of [1], $(\lambda I - T)_{|M\cap D(T)}$ remains semi-regular for $|\lambda| < \gamma\left(T_{|M\cap D(T)}\right)$. As $T_{|N}$ is quasinilpotent, for all $\lambda \in \mathbb{C}\diagdown\{0\}$ we know that $(\lambda I - T)_{|N}$ is invertible, then $(\lambda I - T)_{|N}$ is semi-regular. Finally, the semi-regularity of $(\lambda I - T)_{|M\cap D(T)}$ and $(\lambda I - T)_{|N}$ implies that of $(\lambda I - T)$ for all $\lambda \in D(0,\epsilon)\diagdown\{0\}$ with $\epsilon = \gamma\left(T_{|M\cap D(T)}\right)$. □

As a straightforward consequence of Theorem 2.6, we have:

Corollary 2.7. *Let $T \in \mathcal{C}(X)$. Then, $\{\lambda \in \mathbb{C} : \lambda I - T$ is pseudo-Fredholm$\}$ is a discrete set in the semi-regular resolvent set $\{\lambda \in \mathbb{C} : \lambda I - T$ is semi-regular$\}$.*

This result is proved only for quasi-Fredholm operators on Hilbert spaces (see Proposition 4.3.1 (a), [14]) but its proof extends without modification to the more general case of the closed pseudo-Fredholm operators on a Banach space. The following results give some characterizations of closed operators having empty pseudo-Fredholm essential spectrum.

Theorem 2.8. *Let $T \in \mathcal{C}(X)$. If $\sigma_{pF}(T)$ is empty, then there exists an infinite sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$, all disctinct, such that $\sigma(T) = \overline{\{\lambda_n : n \in \mathbb{N}\}}$. In particular, $\sigma(T)$ is finite when $T \in \mathcal{B}(X)$ and $\sigma_{pF}(T) = \emptyset$.*

Proof. If $\sigma_{pF}(T) = \emptyset$, $(\lambda I - T)$ is a pseudo-Fredholm operator on X for all $\lambda \in \mathbb{C}$. By virtue of Corollary 2.7, the points λ such that $(\lambda I - T)$ is pseudo-Fredholm are isolated points in the semi-regular resolvent set of T. On the other hand, outside the semi-regular resolvent set of T, we know that the set:

$$\{\lambda \in \mathbb{C} : (\lambda I - T) \text{ is not semi-regular}\} \diagdown \sigma_{pF}(T)$$

is at most countable ([7]). Hence $\sigma(T)$ consist of at most countably many isolated points. Furthermore, if $T \in \mathcal{B}(X)$, then $\sigma(T)$ is a discrete compact subset of $\mathbb{C}$, and thus it is necessary finite. □

Remark 2.9. 1) If $\dim X < \infty$, then $(\lambda I - T)$ is pseudo-Fredholm for all $\lambda \in \mathbb{C}$, and hence $\sigma_{pF}(T) = \emptyset$.

2) If $\sigma_{qF}(T) = \{\lambda_0\}$, then there exists an infinite sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}^*}$, all disctinct, such that $\sigma(T) = \overline{\{\lambda_n : n \in \mathbb{N}^*\}} \cup \{\lambda_0\}$. Moreover two cases can occur :

a) the sequence (λ_n) is finite.

b) the sequence (λ_n) is infinite and $\lim\limits_{n\to+\infty} \lambda_n = \lambda_0$.

Proposition 2.10. *Let $T \in \mathcal{C}(X)$. If $\sigma(T)$ is a discrete subset of $\mathbb{C}$, then $\sigma_{pF}(T)$ is empty.*

Proof. If $\sigma(T)$ is discrete, then each point of the spectrum of T is isolated. Therefore, for every $\lambda \in \sigma(T)$, $K(\lambda I - T)$ and $H_0(\lambda I - T)$ are closed, T-invariant subspaces of X, $X = K(\lambda I - T) \oplus H_0(\lambda I - T)$ and where $T_{|K(\lambda I - T)}$ is semi-regular and $T_{|H_0(\lambda I - T)}$ is quasinilpotent, thus $\lambda I - T$ is pseudo-Fredholm. It follows then that $\sigma_{pF}(T) = \emptyset$. □

We achieve this section with a result of Jordanization type of operators whose quasi-Fredholm spectrum is empty on infinite-dimensional Banach spaces.

Theorem 2.11. *([Proposition 5.1.2, [14]]) Let $T \in \mathcal{B}(H)$, H infinite-dimensional Hilbert space, such that $\sigma_{qF}(T)$ is empty. Then:*

1) there exists closed subspaces N_k, $k \in \{1, ..., n\}$, of H such that $H = \bigoplus\limits_{k=1}^{n} N_k$, $T(N_k) \subseteq N_k$ and $(\lambda_k I - T)_{|N_k}$ is nilpotent, for all k.

2) there exists a nilpotent operator Q on H and $S = \sum\limits_{k=1}^{n} \lambda_k P_k$, P_k is the projection of H on N_k corresponding to the decomposition $H = \bigoplus\limits_{k=1}^{n} N_k$, such that $T = S + Q$ and $SQ = QS$.

Remark 2.12. ([Propositions 5.5.2 and 5.5.3, [14]]). If $\sigma_{qF}(T) = \{\lambda_0\}$, then:

A) in the case where the sequence (λ_n) is finite, there exists closed subspaces N_k, $k \in \{0, ..., n\}$, of H such that $H = \bigoplus_{k=0}^{n} N_k$, $T(N_k) \subseteq N_k$, $Q_0 = (\lambda_0 I - T)_{|N_0}$ is quasinilpotent and $Q_k = (\lambda_k I - T)_{|N_k}$ is nilpotent, for all $k \in \{1, ..., n\}$. Let $Q = \sum_{k=0}^{n} Q_k P_k$ and $S = \sum_{k=0}^{n} \lambda_k P_k$, where P_k is the projection of H on N_k corresponding to the decomposition $H = \bigoplus_{k=0}^{n} N_k$. Then $T = S + Q$, Q is quasinilpotent and $SQ = QS$.

B) in the case where the sequence (λ_n) is infinite, we suppose in addition that T is normal on H. Then there exists closed subspaces $\{N_k\}_{k\in\mathbb{N}}$ of H, mutually orthogonal, such that $H = \bigoplus_{k=0}^{\infty} N_k$, $T(N_k) \subseteq N_k$, $Q_0 = (\lambda_0 I - T)_{|N_0}$ is quasinilpotent and $Q_k = (\lambda_k I - T)_{|N_k}$ is nilpotent, for all $k \in \mathbb{N}^*$. Let $Q = \sum_{k=0}^{\infty} Q_k P_k$ and $S = \sum_{k=0}^{\infty} \lambda_k P_k$, where P_k is the orthogonal projection of H on N_k corresponding to the decomposition $H = \bigoplus_{k=0}^{\infty} N_k$. Then $T = S + Q$, S is normal, Q is quasinilpotent and $SQ = QS$.

3. Applications

3.1. **Operator Matrices with finite spectrum.** Let X, Y, Z be Banach spaces and $Z = X \oplus Y$. For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ consider the operator:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{B}(Z).$$

It is well know that, in the case of infinite dimensional, $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, this shows that $\sigma(M_C)$ is finite as soon as $\sigma(A)$ and $\sigma(A)$ are finite subsets of $\mathbb{C}$. We can directly deduce a weaker condition so that $\sigma(M_C)$ is finite by using Theorem 4.1 of [20].

Theorem 3.1. *If* $\max\{d(\lambda I - A), d(\lambda I - B), d(\lambda I - A^*), d(\lambda I - B^*)\} < \infty$ *for every* $\lambda \in \mathbb{C}$, *then* $\sigma_{pF}(M_C) = \sigma_{pF}(A) \cup \sigma_{pF}(B)$. *Furthermore, if* $\sigma_{pF}(A)$ *and* $\sigma_{pF}(B)$ *are empty, then* $\sigma(M_C)$ *is finite.*

3.2. **Totally paranormal operators.** A bounded linear operator T on a Banch space X is said to be paranormal if:

$$\|Tx\|^2 \leq \left\|T^2 x\right\| \|x\|, \text{ for all } x \in X.$$

Note that every paranormal operator T is normaloid (i.e., $\|T\| = r(T)$ the spectral radius of T) or, equivalently, $\|T^n\| = \|T\|^n$ for every $n \in \mathbb{N}$. So, if T is a quasinilpotent paranormal operator, then $T = 0$. It is easy to see that the restriction of T to every closed invariant subspace is also paranormal. It is known that the property of being paranormal is not translation-invariant by scalars. Paranormal operators do not satisfy $H_0(\lambda I - T) = N((\lambda I - T)^p)$, for some integer $p \geq 1$ and every $\lambda \in \mathbb{C}$. However, isolated points of the spectrum of a paranormal operator T are simple poles of the resolvent $R(\lambda, T)$ of T (see [2]). The definition of paranormality of T implies that $a(T) \leq 1$. $T \in \mathcal{B}(X)$ is said to be totally paranormal if $(\lambda I - T)$ is paranormal for every $\lambda \in \mathbb{C}$. Examples of totally paranormal operators are all the operators T on a Hilbert space H which are hyponormal, $\|T^* x\| \leq \|Tx\|$, for all $x \in H$. Thus, every totally paranormal operator has ascent $a(\lambda I - T) \leq 1$ for every $\lambda \in \mathbb{C}$. By an easy inductive argument we see that $\|(\lambda I - T)\,x\|^n \leq \|(\lambda I - T)^n\,x\|\,\|x\|^{n-1}$, for all $x \in X$, $\lambda \in \mathbb{C}$ and $n \geq 1$. Hence, $H_0(\lambda I - T) = N(\lambda I - T)$, for every $\lambda \in \mathbb{C}$. Furthermore, every isolated points of $\sigma(T)$ is an eigenvalue of T.

Theorem 3.2. *Let $T \in \mathcal{B}(H)$ where H is an infinite-dimensional Hilbert space.*

1) If T is paranormal such that $H_0(T)$ and $H_0(T) + R(T)$ are closed, then T is a pseudo-Fredholm operator.

2) If T is totally paranormal such that $H_0(\lambda I - T)$ and $H_0(\lambda I - T) + R(\lambda I - T)$ are closed for every $\lambda \in \mathbb{C}$, then the spectrum $\sigma(T)$ of T is finite.

Proof. 1) Consider $M = H_0(T)^{\perp}$, then $H = M \oplus H_0(T)$ and $T_{|M}$ is injective with closed range (see [10]). Thus, $T_{|M}$ is semi-regular. We have $H_0(T) = H_0(T_{|H_0(T)})$ and by virtue of [Theorem 1.5, [21]] we obtain that $T_{|H_0(T)}$ is quasinilpotent.

2) Using (1), we see that $(\lambda I - T)$ is pseudo-Fredholm for every $\lambda \in \mathbb{C}$. So $\sigma_{pF} = \emptyset$ and by Theorem 2.8, $\sigma(T)$ is necessarily a finite subset of $\mathbb{C}$. □

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[1]Department of Mathematics, University of Mostaganem, Faculty of Sciences, Mostaganem, Algeria. Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO).

E-mail address: messirdi.sofiane@hotmail.fr

[2]Department of Mathematics, University of Mostaganem, Faculty of Sciences, Mostaganem, Algeria.

E-mail address: mohand_ouldalidz@yahoo.fr

[3]Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO).

E-mail address: bmessirdi@yahoo.fr